Gauss-Legendre integration

Numerical integration in SMath (v0.85) for a noncontinous function will fail:

$$f(x) := \text{if } x > 0 \quad x^3 \quad \text{else} \quad x^2$$

$a := -6 \qquad b := 5$

$$\int_a^b f(x)\, dx = \blacksquare$$

Let us solve this example by adding two integrals:

$$\int_a^b f(x)\, dx = \int_a^0 x^2\, dx + \int_0^b x^3\, dx$$

It is simple to solve it simbolically. The result is:

$$-\frac{a^3}{3} + \frac{b^4}{4} = 228.25$$

We can use Gauss-Legendre numerical method of integration in this case. This could also be applied for continous functions as well. In this example 10 node points will be used and m subintervals.

Abscissas Weights

$$\xi := \begin{pmatrix} -0.973906528517 \\ -0.865063366689 \\ -0.679409568299 \\ -0.433395394129 \\ -0.148874338982 \\ 0.148874338982 \\ 0.433395394129 \\ 0.679409568299 \\ 0.865063366689 \\ 0.973906528517 \end{pmatrix} \qquad w := \begin{pmatrix} 0.066671344308 \\ 0.149451349151 \\ 0.219086362516 \\ 0.26926671931 \\ 0.295524224715 \\ 0.295524224715 \\ 0.26926671931 \\ 0.219086362516 \\ 0.149451349151 \\ 0.066671344308 \end{pmatrix} \qquad n := \text{length}(\xi)$$

Interval of integration:

$a = -6 \qquad b = 5$

$m := 8$ Number of subintervals. Increase it (by step 1) in order to obtain the desired accuracy.

$$h := \frac{b-a}{m}$$

$$x := a, a+h .. b$$

$$ans := \sum_{j=1}^{m} \left(\frac{x_{j+1} - x_j}{2} \cdot \sum_{k=1}^{n} \left(w_k \cdot f\left(\frac{x_{j+1} - x_j}{2} \cdot \xi_k + \frac{x_{j+1} + x_j}{2} \right) \right) \right)$$

$ans = 228.25$

Here is another function:

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{else} \end{cases}$$

$a := -6 \qquad b := 5$

$$\int_a^b f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^b f(x)\,dx = \int_a^0 -1\,dx + \int_0^b 1\,dx$$

Result:

Symbolically : $-(-a) + b = -1$

Numerically : $\int_a^0 -1\,dx + \int_0^b 1\,dx = -1$

Using Gauss-Legendre Integration :

$m := 25$ Number of subintervals. Increase it (by step 1) in order to obtain the desired accuracy.

$$h := \frac{b-a}{m}$$

$x := a, a+h .. b$

$$ans := \sum_{j=1}^{m} \left[\frac{x_{j+1} - x_j}{2} \cdot \sum_{k=1}^{n} \left[w_k \cdot f\left(\frac{x_{j+1} - x_j}{2} \cdot \xi_k + \frac{x_{j+1} + x_j}{2} \right) \right] \right]$$

$ans = -1.01$

Continous function:

$f(x) := \sin(x) + \cos(x)$

$a := -2 \qquad b := 3$

$$\int_a^b f(x)\,dx = 1.6243$$

$$-\cos(b) + \sin(b) + \cos(a) - \sin(a) = 1.6243$$

$m := 1$ Number of subintervals. Increase it (by step 1) in order to obtain the desired accuracy.

$$h := \frac{b-a}{m}$$

$x := a, a+h .. b$

$$ans := \sum_{j=1}^{m} \left[\frac{x_{j+1} - x_j}{2} \cdot \sum_{k=1}^{n} \left[w_k \cdot f\left(\frac{x_{j+1} - x_j}{2} \cdot \xi_k + \frac{x_{j+1} + x_j}{2} \right) \right] \right]$$

$ans = 1.6243$