$y\left(x, a, x_U, y_U\right) := a \cdot \cosh\left(\frac{x - x_U}{a}\right) - a + y_U$ $\quad y'\left(x, a, x_U\right) := \sinh\left(\frac{x - x_U}{a}\right)$

$arcsinh(x) := \operatorname{asinh}(x)$ For maple outputs

$$L_c\left(x_1, x_2, a, x_U\right) := \operatorname{maple}\left(\int_{x_1}^{x_2} \sqrt{1 + \left(y'\left(x, a, x_U\right)\right)^2}\, dx\right)$$

$$x_{cg}\left(x_1, x_2, a, x_U\right) := \operatorname{maple}\left(\frac{\int_{x_1}^{x_2} x \cdot \sqrt{1 + \left(y'\left(x, a, x_U\right)\right)^2}\, dx}{L_c\left(x_1, x_2, a, x_U\right)}\right)$$

$$y_{cg}\left(x_1, x_2, a, x_U, y_U\right) := \operatorname{maple}\left(\frac{\int_{x_1}^{x_2} y\left(x, a, x_U, y_U\right) \cdot \sqrt{1 + \left(y'\left(x, a, x_U\right)\right)^2}\, dx}{L_c\left(x_1, x_2, a, x_U\right)}\right)$$

$$PE\left(a, x_{UL}, y_{UL}, x_{UR}, y_{Ur}\right) := \operatorname{eval}\left(\sum \begin{bmatrix} m_c\ g_e \cdot L_c\left(0\ \mathrm{m}, x, a, x_{UL}\right) \cdot y_{cg}\left(0\ \mathrm{m}, x, a, x_{UL}, y_{UL}\right) \\ m\ g_e \cdot y\left(x, a, x_{UL}, y_{UL}\right) \\ m_c\ g_e \cdot L_c\left(x, L, a, x_{UR}\right) \cdot y_{cg}\left(x, L, a, x_{UR}, y_{UR}\right) \end{bmatrix}\right)$$

$L := 6\ \mathrm{m}$ $\quad S := 12\ \mathrm{m}$ $\quad H_L := 7\ \mathrm{m}$ $\quad H_R := 5\ \mathrm{m}$ $\quad m := 0.1\ \mathrm{kg}$ $\quad m_c := 0.1\ \frac{\mathrm{kg}}{\mathrm{m}}$ $\quad x := 4\ \mathrm{m}$

Lagrange's multipliers method function

$$LM := \frac{1}{\mathrm{m}} \cdot \begin{bmatrix} S - \left(L_c\left(0\ \mathrm{m}, x, a, x_{UL}\right) + L_c\left(x, L, a, x_{UR}\right)\right) \\ H_L - y\left(0\ \mathrm{m}, a, x_{UL}, y_{UL}\right) \\ y\left(L, a, x_{UR}, y_{UR}\right) - H_R \\ y\left(x, a, x_{UL}, y_{UL}\right) - y\left(x, a, x_{UR}, y_{UR}\right) \end{bmatrix}$$

Lagrange's "G" function

$$L\left(u_i, \lambda_j\right) = F - \left(\sum \left(\lambda_j \cdot G_j\right)\right)$$

$$LM(u\#) := \left|\begin{array}{l} \left[a\ x_{UL}\ y_{UL}\ x_{UR}\ y_{UR}\ \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\right] := u\#^{\mathrm{T}} \\ \operatorname{eval}\left(\frac{1}{\mathrm{J}} \cdot PE\left(a, x_{UL}, y_{UL}, x_{UR}, y_{Ur}\right) - \left[\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\right]^{\mathrm{T}} \cdot LM\right) \end{array}\right.$$

Target function: the transposed gradient

$r\# := [1..9]$

$$\varphi(u\#) := \left|\begin{array}{l} \left[a\# := un\# \cdot u\#\ \ G\# := 0\ \ LMa\# := \operatorname{eval}\left(LM(a\#)\right)\right] \\ G\#_{r\#} := \left(LM\left(a\# + \operatorname{col}(h\#, r\#)\right) - LMa\#\right) \cdot \frac{\operatorname{UnitsOf}\left(h\#_{r\#\, r\#}\right)}{h\#_{r\#\, r\#}} \end{array}\right.$$

Numerical solver

$uo\# := \left[2\ \mathrm{m}\ 2\ \mathrm{m}\ 2\ \mathrm{m}\ 2\ \mathrm{m}\ 2\ \mathrm{m}\ 1\ 1\ 1\ 1\right]^{\mathrm{T}}$ guess value

$un\# := \operatorname{diag}\left(\overrightarrow{\operatorname{UnitsOf}(uo\#)}\right)$ $\quad h\# := 10^{-5} \cdot un\#$ derivatives forward step

$\begin{bmatrix} a & x_{UL} & y_{UL} & x_{UR} & y_{UR} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix} := un\# \cdot \text{al\_nleqsolve}\left(\overrightarrow{\frac{uo\#}{\text{UnitsOf}(uo\#)}}, \varphi\right)$

$a = 1.8041\ \text{m}$ $x_{UL} = 3.834\ \text{m}$ $y_{UL} = 1.1423\ \text{m}$

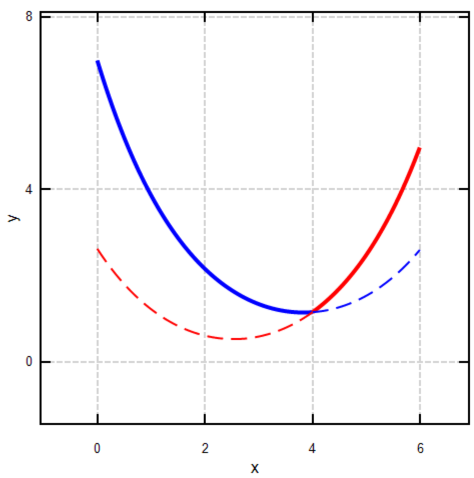
$x_{UR} = 2.5377\ \text{m}$ $y_{UR} = 0.5241\ \text{m}$

$PE\left(a, x_{UL}, y_{UL}, x_{UR}, y_{Ur}\right) = 40.6931\ \text{J}$

Sanitiy check: $\text{normi}(LM) = 1.784 \cdot 10^{-8}$

$x_L := \left[(0\ \text{m}), \frac{x}{300} .. x\right]$ $x_R := \left[x, x + \frac{L - x}{300} .. L\right]$

$$Plot := \begin{cases} \text{augment}\left(\frac{x_L}{\text{m}}, \overrightarrow{\frac{y(x_L, a, x_{UL}, y_{UL})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_R}{\text{m}}, \overrightarrow{\frac{y(x_R, a, x_{UR}, y_{UR})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_R}{\text{m}}, \overrightarrow{\frac{y(x_R, a, x_{UL}, y_{UL})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_L}{\text{m}}, \overrightarrow{\frac{y(x_L, a, x_{UR}, y_{UR})}{\text{m}}}\right) \end{cases}$$




Plot

$a := 2.5614407 \frac{\text{N}}{\text{N m}^{-1}}$ $x_{UL} := 4.4116127\ \text{m}$ $y_{UL} := 2.163828\ \text{m}$

$x_{UR} := 2.1353467\ \text{m}$ $y_{UR} := 1.4877577\ \text{m}$ Found in Mathcad

$PE\left(a, x_{UL}, y_{UL}, x_{UR}, y_{Ur}\right) = 41.275\ \text{J}$

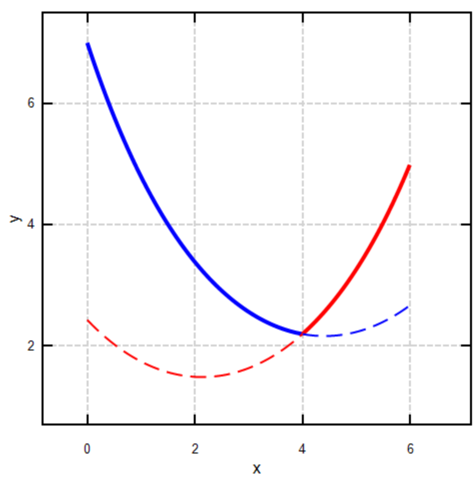
Sanitiy check: $\text{normi}(LM) = 2$

$$LM = \begin{bmatrix} 2 \\ 7.6057 \cdot 10^{-8} \\ -5.7498 \cdot 10^{-8} \\ 6.7884 \cdot 10^{-8} \end{bmatrix}$$

Not verify this equation:

$S - \left(L_c\left(0\ \text{m}, x, a, x_{UL}\right) + L_c\left(x, L, a, x_{UR}\right)\right) = 2\ \text{m}$

$$Plot := \begin{cases} \text{augment}\left(\frac{x_L}{\text{m}}, \overrightarrow{\frac{y(x_L, a, x_{UL}, y_{UL})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_R}{\text{m}}, \overrightarrow{\frac{y(x_R, a, x_{UR}, y_{UR})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_R}{\text{m}}, \overrightarrow{\frac{y(x_R, a, x_{UL}, y_{UL})}{\text{m}}}\right) \\ \text{augment}\left(\frac{x_L}{\text{m}}, \overrightarrow{\frac{y(x_L, a, x_{UR}, y_{UR})}{\text{m}}}\right) \end{cases}$$




Plot